Mathematical tools for balanced bases

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1 Standard bases

Before I get into balanced forms, it will be necessary to highlight the relevant parts of number theory for standard bases, this is really just 3:nd grade mathematics with fancy symbols, but the definitions are necessary later.

For any standard base b, any given number n can be expressed as a sum of N-1 digits a_k in the range $0 \leq a_k < b$

$$n = \sum_{k=0}^{N-1} b^k a_k \tag{1}$$

For a standard base, for any $\kappa \in \mathbb{N}_0$ the following is true

$$mod_{b^\kappa} n = \sum_{k=0}^{\kappa-1} b^k a_k \tag{2}$$

where $mod_m n$ is the remainder of $\frac{n}{m}$, or in other words for positive n and m,

$$mod_m n = n - m \left\lfloor \frac{n}{m} \right\rfloor \tag{3}$$

Already having the remainder, bringing the quotient into the game makes perfect sense. From the relationship

$$n = m \cdot quot_m n + mod_m n \tag{4}$$

The quotient is found to be

$$quot_m n = \frac{n - mod_m n}{m} = \left\lfloor \frac{n}{m} \right\rfloor \tag{5}$$

But looking back at (2), the quotient can now be expressed as

$$quot_{b^\kappa} n = b^{-\kappa} \sum_{k=\kappa}^{N-1} b^k a_k \tag{6}$$

2 Balanced form

Now, to do the same for balanced bases. For any odd balanced base b, any given number n can be expressed as a sum of N-1 digits a_k in the range $-\lfloor \frac{b}{2} \rfloor \leq a_k \leq \lfloor \frac{b}{2} \rfloor$.

$$n = \sum_{k=0}^{N-1} b^k a_k \tag{7}$$

If you the previously defined mod and quot operators, you will get the remainder and quotient of the standard base b, and not the balanced base. It is necessary to modify them to get an operator that satisfies the following:

$$R_{b^\kappa}(n) = \sum_{k=0}^{\kappa-1} b^k a_k \tag{8}$$

Empirically, the following operator satisfies

$$R_{b^\kappa}(n) = \begin{cases} mod_{b^\kappa}\left(n + \lfloor \frac{b^\kappa}{2} \rfloor\right) - \lfloor \frac{b^\kappa}{2} \rfloor & n \geq 0 \\ -mod_{b^\kappa}\left(-n + \lfloor \frac{b^\kappa}{2} \rfloor\right) + \lfloor \frac{b^\kappa}{2} \rfloor & n < 0 \end{cases} \tag{9}$$

From the above definition of $mod_m n$, R can be expressed as

$$R_{b^\kappa}(n) = \begin{cases} n - b^\kappa \left\lfloor \frac{n + \lfloor \frac{b^\kappa}{2} \rfloor}{b^\kappa} \right\rfloor & n \geq 0 \\ n + b^\kappa \left\lfloor \frac{-n + \lfloor \frac{b^\kappa}{2} \rfloor}{b^\kappa} \right\rfloor & n < 0 \end{cases} \tag{10}$$

But since b is positive and even, $\lfloor \frac{b^\kappa}{2} \rfloor = \frac{b^\kappa - 1}{2}$, so

$$R_{b^\kappa}(n) = \begin{cases} n - b^\kappa \lfloor \frac{2n + b^\kappa - 1}{2b^\kappa} \rfloor & n \geq 0 \\ n + b^\kappa \lfloor \frac{-2n + b^\kappa - 1}{2b^\kappa} \rfloor & n < 0 \end{cases} \tag{11}$$

Using the equivalent relationship as previously for the standard quotient, i.e.

$$n = m \cdot Q_m n + R_m n \tag{12}$$

Q is easily solved and found to be

$$Q_{b^\kappa}(n) = \begin{cases} \lfloor \frac{2n + b^\kappa - 1}{2b^\kappa} \rfloor & n \geq 0 \\ -\lfloor \frac{-2n + b^\kappa - 1}{2b^\kappa} \rfloor & n < 0 \end{cases} = \sum_{k=\kappa}^{N-1} b^{k-\kappa} a_k \tag{13}$$

A very useful and directily obvious relationship is that

$$a_\kappa = Q_{b^\kappa}(n) - bQ_{b^{\kappa+1}}(n) \tag{14}$$

In other words

$$a_\kappa = \begin{cases} \lfloor \frac{2n + b^\kappa - 1}{2b^\kappa} \rfloor - b \left\lfloor \frac{2n + b^{\kappa+1} - 1}{2b^{\kappa+1}} \right\rfloor & n \geq 0 \\ b \left\lfloor \frac{-2n + b^{\kappa+1} - 1}{2b^{\kappa+1}} \right\rfloor - \lfloor \frac{-2n + b^\kappa - 1}{2b^\kappa} \rfloor & n < 0 \end{cases} \tag{15}$$